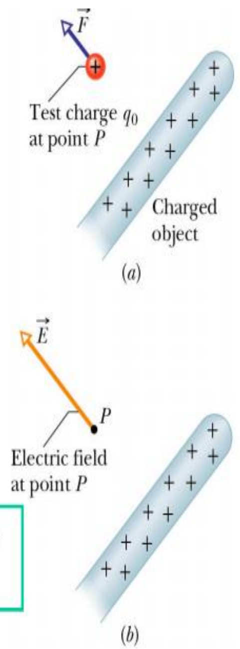
Electricity & magnetism-1

Electric field

Field E is defined as the force that would be felt by a unit positive test charge

$$\vec{E} = \vec{F} / q_0$$

SI units for the electric field: newtons per coulomb.

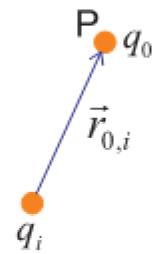


Principle of Superposition:

In a system with N charges, the **total** E-field due to all charges is the **vector sum** of E-field due to individual charges.

i.e. $\boxed{\vec{E} = \sum_i \vec{E}_i = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{r_i^2} \hat{r}_i}$

(a) E-field due to a single charge q_i:



From the definitions of **Coulomb's Law**, the force experienced at location of q_0 (point P)

$$\vec{F}_{0,i} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_0 q_i}{r_{0,i}^2} \cdot \hat{r}_{0,i}$$

where $\hat{r}_{0,i}$ is the unit vector along the direction *from charge* q_i *to* q_0,

$$\begin{aligned} \hat{r}_{0,i} &= \text{Unit vector from charge } q_i \text{ to point P} \\ &= \hat{r}_i \text{ (radical unit vector from } q_i\text{)} \end{aligned}$$

Recall $\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$

$\therefore$ E-field due to q_i at point P:

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_i}{r_i^2} \cdot \hat{r}_i$$

where $\vec{r}_i$ = Vector pointing from q_i to point P,
thus $\hat{r}_i$ = Unit vector pointing from q_i to point P
Note:

(1) E-field is a **vector**.

(2) Direction of E-field depends on **both** position of P and sign of q_i.

Electric Field Lines

We **visualize** the field by drawing **field lines.**

These are defined by **three properties:**

- Lines point in the **same direction** as the field.
- Density of lines gives the **magnitude** of the field.
- Lines begin on + charges; end on – charges.

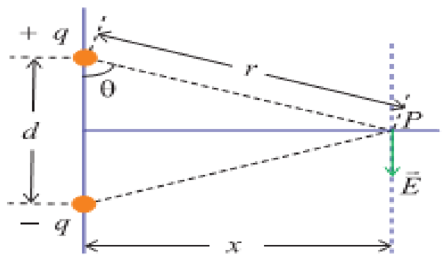
Electric Dipole

- The combination of two charges of equal but opposite sign is called a dipole.
- If the charges +q and –q are separated by a distance d, then the ***dipole moment*** $\vec{p}$ is defined as a vector pointing from –q to +q of magnitude $p = qd$.

$$-q \quad \vec{p} \quad +q$$

Electric Field due to dipole

Magnitude of E-field $= 2E_+ \cos\theta$



$$\therefore E = 2\Big(\overbrace{\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}}^{E_+ \text{ or } E_- \text{ magnitude!}}\Big)\cos\theta$$

But $\quad r = \sqrt{\left(\frac{d}{2}\right)^2 + x^2}$

$$\cos\theta = \frac{d/2}{r}$$

$$\therefore E = \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{[x^2+(\frac{d}{2})^2]^{\frac{3}{2}}}$$

$$(p = qd)$$

<u>Special case:</u> When $x \gg d$

$$[x^2+(\frac{d}{2})^2]^{\frac{3}{2}} = x^3[1+(\frac{d}{2x})^2]^{\frac{3}{2}}$$

- Binomial Approximation:

$$(1+y)^n \approx 1+ny \qquad \text{if } y \ll 1$$

$$\boxed{\text{E-field of dipole} \doteqdot \frac{1}{4\pi\epsilon_0}\cdot\frac{p}{x^3} \sim \frac{1}{x^3}}$$

- Compare with $\frac{1}{r^2}$ E-field for single charge
- Result also valid for point P along any axis with respect to dipole

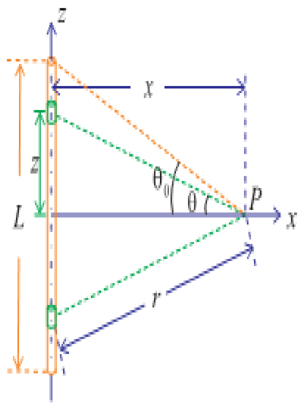
Electric field due to line of Charge

(1) Symmetry considered: The E-field from $+z$ and $-z$ directions *cancel along z-direction*, $\therefore$ Only horizontal E-field components need to be considered.

(2) For each element of length dz, charge $dq = \lambda dz$

$\therefore$ Horizontal E-field at point P due to element dz =

$$|d\vec{E}|\cos\theta = \underbrace{\frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dz}{r^2}}_{dE_{dz}}\cos\theta$$



$\therefore$ E-field due to entire line charge at point P

$$E = \int_{-L/2}^{L/2}\frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dz}{r^2}\cos\theta$$

$$= 2\int_0^{L/2}\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{dz}{r^2}\cos\theta$$

To calculate this integral:

- First, notice that x is fixed, but z, r, θ all varies.
- Change of variable (from z to θ)

(1) $z = x\tan\theta \quad \therefore dz = x\sec^2\theta\, d\theta$
$x = r\cos\theta \quad \therefore r^2 = x^2\sec^2\theta$

(2) When $z = 0\ ,\quad \theta = 0°$
$z = L/2 \quad \theta = \theta_0 \quad$ where $\tan\theta_0 = \frac{L/2}{x}$

$$\begin{aligned} E &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{x\sec^2\theta\, d\theta}{x^2\sec^2\theta}\cdot\cos\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\int_0^{\theta_0}\frac{1}{x}\cdot\cos\theta\, d\theta \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot(\sin\theta)\Big|_0^{\theta_0} \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\sin\theta_0 \\ &= 2\cdot\frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{x}\cdot\frac{L/2}{\sqrt{x^2+(\frac{L}{2})^2}} \end{aligned}$$

$$\boxed{E = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x\sqrt{x^2+(\frac{L}{2})^2}}}$$ along x-direction

<u>Important limiting cases:</u>

1. $x \gg L$: $\quad E \doteqdot \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x^2}$
   But λL = Total charge on rod
   $\therefore$ System behave like a point charge
2. $L \gg x$: $\quad E \doteqdot \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda L}{x\cdot\frac{L}{2}}$

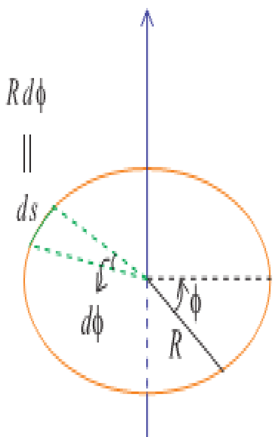
$$\boxed{E_x = \frac{\lambda}{2\pi\epsilon_0 x}}$$

Electric field due to ring of charge

(1) Symmetry considered: For every charge element dq considered, there exists $dq\prime$ where the horizontal $\vec{E}$ field components cancel.

$\Rightarrow$ Overall E-field lies along z-direction.

(2) For each element of length dz, charge



$360^{\circ} = 2\pi$ radian

$180^{\circ} = \pi$ radian

$$dq = \underset{\substack{\text{Linear}\\ \text{charge density}}}{\underset{\uparrow}{\lambda}} \cdot \underset{\substack{\text{Circular}\\ \text{length element}}}{\underset{\uparrow}{ds}}$$

$dq = \lambda \cdot R\, d\phi,$ where ϕ is the angle measured on the ring plane

$\therefore$ Net E-field along z-axis due to dq:

$$dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2} \cdot \cos\theta$$

$$\text{Total E-field} = \int dE$$

$$= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R\, d\phi}{r^2} \cdot \cos\theta \qquad \left(\cos\theta = \frac{z}{r}\right)$$

Note: Here in this case, θ, R and r are *fixed* as ϕ varies! BUT we want to convert r, θ to R, z.

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda R z}{r^3} \int_0^{2\pi} d\phi$$

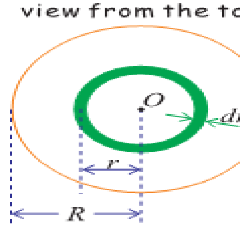
$$\boxed{E = \frac{1}{4\pi\epsilon_0} \cdot \frac{\lambda(2\pi R) z}{(z^2 + R^2)^{3/2}}} \qquad \text{along } z\text{-axis}$$

BUT: $\lambda(2\pi R)$ = total charge on the ring

Electric field due to disk of charge

Total charge of ring

$$dq = \sigma \cdot (\underbrace{2\pi r\ dr}_{\text{Area of the ring}})$$



Recall from Example 2:

$$\text{E-field from ring: } dE = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq\ z}{(z^2+r^2)^{3/2}}$$

$$\begin{aligned} \therefore \quad E &= \frac{1}{4\pi\epsilon_0} \int_0^R \frac{2\pi\sigma r\ dr \cdot z}{(z^2+r^2)^{3/2}} \\ &= \frac{1}{4\pi\epsilon_0} \int_0^R 2\pi\sigma z \frac{r\ dr}{(z^2+r^2)^{3/2}} \end{aligned}$$

- Change of variable:

$$\begin{aligned} & u = z^2 + r^2 \quad \Rightarrow \quad (z^2+r^2)^{3/2} = u^{3/2} \\ \Rightarrow \quad & du = 2r\ dr \quad \Rightarrow \quad r\ dr = \tfrac{1}{2} du \end{aligned}$$

- Change of integration limit:

$$\begin{cases} r = 0 \quad , \quad u = z^2 \\ r = R \quad , \quad u = z^2 + R^2 \end{cases}$$

$$\therefore\ E = \frac{1}{4\pi\epsilon_0} \cdot 2\pi\sigma z \int_{z^2}^{z^2+R^2} \frac{1}{2}\, u^{-3/2} du$$

BUT: $\int u^{-3/2} du = \frac{u^{-1/2}}{-1/2} = -2u^{-1/2}$

$$\begin{aligned} \therefore\ E &= \frac{1}{2\epsilon_0}\, \sigma z\, \left(-u^{-1/2}\right)\Big|_{z^2}^{z^2+R^2} \\ &= \frac{1}{2\epsilon_0}\, \sigma z\, \left(\frac{-1}{\sqrt{z^2+R^2}} + \frac{1}{z}\right) \end{aligned}$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]}$$

<u>VERY IMPORTANT LIMITING CASE:</u>

If $R \gg z$, that is if we have an <u>infinite sheet of charge</u> with charge density σ:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] \\ &\simeq \frac{\sigma}{2\epsilon_0}\left[1 - \frac{z}{R}\right] \end{aligned}$$

$$\boxed{E \approx \frac{\sigma}{2\epsilon_0}}$$

E-field is normal to the charged surface

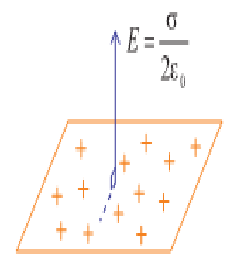


Figure 2.2: E-field due to an infinite sheet of charge, charge density = σ

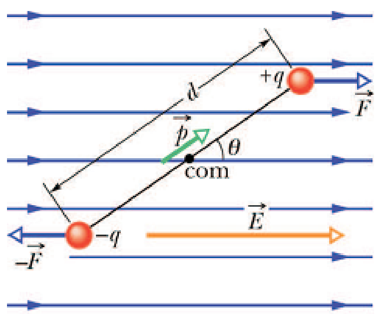
Dipole in electric field

Assumption: E-field from dipole doesn't affect the external E-field.

- Dipole moment:

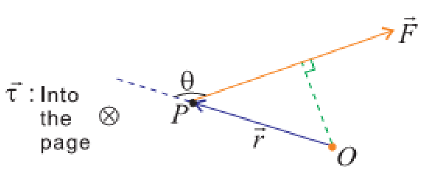
$$\vec{p} = q\vec{d}$$

- Force due to the E-field on $+ve$ and $-ve$ charge are *equal and opposite in direction*. Total external force on dipole = 0.

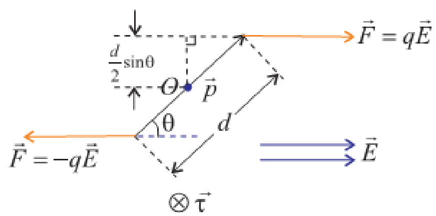


$\boxed{\text{BUT: There is an external \textbf{torque} on the center of the dipole.}}$

Reminder:



Force $\vec{F}$ exerts at point P. The force exerts a **torque** $\vec{\tau} = \vec{r} \times \vec{F}$ on point P with respect to point O.



- direction: clockwise torque
- magnitude:

$$\begin{aligned}\tau &= \tau_{+ve} + \tau_{-ve} \\ &= F \cdot \frac{d}{2}\sin\theta + F \cdot \frac{d}{2}\sin\theta \\ &= qE \cdot d\sin\theta \\ &= pE\sin\theta\end{aligned}$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

When the dipole $\vec{p}$ rotates $d\theta$, the E-field does work.

Work done by external E-field on the dipole:

$$dW = -\tau\, d\theta$$

Negative sign here because torque by E-field acts to *decrease* θ.

BUT: Because E-field is a **conservative force field** [1] [2], we can define a **potential energy** (U) for the system, so that

$$\boxed{dU = -dW}$$

$\therefore$ For the dipole in external E-field:

$$dU = -dW = pE\sin\theta\, d\theta$$

$$\therefore\ U(\theta) = \int dU = \int pE\sin\theta\, d\theta$$
$$= -pE\cos\theta + U_0$$

Continue….

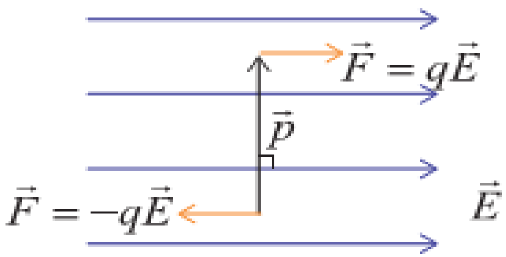
set $U(\theta = 90°) = 0$,

$$\therefore\ 0 = -pE\cos 90° + U_0$$

$$\therefore\ U_0 = 0$$

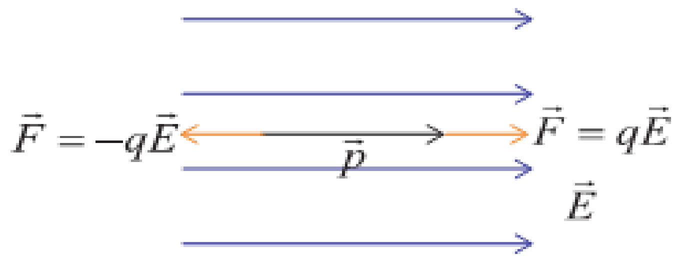
$\therefore$ Potential energy:

$$\boxed{U = -pE\cos\theta = -\vec{p}\cdot\vec{E}}$$



$\theta = 90^\circ$
Torque $|\vec{\tau}| = pE$

$U = 0$ (define)



$\theta = 0^\circ$
Torque $|\vec{\tau}| = 0$

$U = -pE$
(based on definition)

Minimum energy configuration

Exercise

26-1 Ans. $E = F/q = ma/q$. Then

$$E = (9.11\times10^{-31}\text{kg})(1.84\times10^{9}\text{m/s}^2)/(1.60\times10^{-19}\text{C}) = 1.05\times10^{-2}\text{N/C}.$$

26-3 Ans. $F = W$, or $Eq = mg$, so

$$E = \frac{mg}{q} = \frac{(6.64\times10^{-27}\,\text{kg})(9.81\,\text{m/s}^2)}{2(1.60\times10^{-19}\,\text{C})} = 2.03\times10^{-7}\,\text{N/C}.$$

26-5 Ans. Rearrange $E = q/4\pi\epsilon_0 r^2$,

$$q = 4\pi(8.85\times10^{-12}\text{C}^2/\text{N}\cdot\text{m}^2)(0.750\,\text{m})^2(2.30\,\text{N/C}) = 1.44\times10^{-10}\text{C}.$$

26-7 Ans. Use Eq. 26-12 for points along the perpendicular bisector. Then

$$E = \frac{1}{4\pi\epsilon_0}\frac{p}{x^3} = (8.99\times10^{9}\text{N}\cdot\text{m}^2/\text{C}^2)\frac{(3.56\times10^{-29}\,\text{C}\cdot\text{m})}{(25.4\times10^{-9}\,\text{m})^3} = 1.95\times10^{4}\text{N/C}.$$

Students assignment sample problem of ch# 26 and exercise 26-27, 26-29